# *Matrix Summability of Spliced Sequences*

**A Project**

**Submitted by**


## Department of Mathematics,

## Pattamundai College, Pattamundai


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$$\mathbf{G}=\begin{bmatrix}1 & 2\\ 3 & 4\\ 6 & 7\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}$$

$$\mathbf{F+G}=\begin{bmatrix}1 & 2\\ 3 & 4\\ 6 & 7\end{bmatrix}+\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1+1 & 2+0\\ 3+1 & 4+1\\ 6+0 & 7+1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 4 & 5\\ 6 & 8\end{bmatrix}.$$

# Report

A project on "Matrix Summability of Spliced Sequences " was under taken by the students of Department of Mathematics under the guidance of Sri ArabindaPandab, HOD Mathematics.It took two months (Feb & March 2020) to carried out the project. Summability methods deals with methods of constructing generalised sums of series, generalised limits of sequences and values of improper integral; when a sequences or series or integral diverges ordinarily sense, some other methods and generalised provide different summability methods. Since the idea of converges of a sequence or series is used in almost all branches of mathematics, various summation methods has deep applications also. Here effect has been made to explain some summability methods such as spliced sequence.

Students got knowledge of some results of summabilty & Spliced sequences.

Finally, the project was completed and submitted on 16th March 2020.

16/03/20

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# Chapter 1

## *SOME RESULTS ON SUMMABILITY*

Summability methods deals with methods of constructing generalised sums of series, generalised limits of sequences and values of improper integral; when a sequences or series or integral diverges ordinarily sense, some other methods and generalised provide different summability methods. Since the idea of converges of a sequence or series is used in almost all branches of mathematics, various summation methods has deep applications also. Here effect has been made to explain some summability methods such as spliced sequence.

Euler developed the convention that a divergent series $\sum_{n=0}^{\infty} a_n = s$ provided $\sum_{n=0}^{\infty} a_n z^n$ converged to $f(z)$ for small values of z and $f(1) = s$.

In this way,

$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n$ converges for $|z| < 1$ and thus

$$\sum_{n=0}^{\infty}(-1)^n = \frac{1}{1+z}|_{z=1} = \frac{1}{2}$$

Carl Friedrich Gauss introduced the concept of the use of infinite process into mathematical analysis.Augustin-Louis Cauchy formalized ideas concerning convergence and divergence of series. Niels Henrik Abel was a important contributor to the ideas concerning convergence and divergence during the early part of the nineteenth century.

Let us now examine two of the earlier efforts in considering series and sequences which diverge.They are those of Abel and Cesàro method of convergence.And also we define some methods of summability theory and the most important theorem **Silverman-Toeplitz** theorem.

## 1.1 Mathematical Priliminaries

### 1.1.1 Abel Convergence

**Definition 1.1.1.** Let $\{a_n\}_0^\infty$ be a sequence of complex numbers.The sequence $\{a_n\}_0^\infty$ is Abel convergent(written (A) convergent) to $L$,
if $\lim_{x\to 1-}(1-x)\sum_{k=0}^{\infty} a_k x^k = L$ exists (where $\lim_{x\to 1-} f(x) = M$ if given $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$ such that if $1-\delta < x < 1$ then $|f(x) - M| < \epsilon$).

**Definition 1.1.2.** Let $0 < \alpha < \frac{\pi}{2}$.A *Stolz* domain of angle $\alpha$,written as $S(\alpha)$,is the domain $\{\omega : |arg(1-\omega)| < \alpha\} \cap S_1(0)$ where $S_r(\omega) = \{z : |z-\omega| < r\}$.

**Definition 1.1.3.** A series of complex numbers,$\sum_{K=0}^{\infty} a_k$, is $(A)$ convergent to $L$, if the sequence of partial sums, $\{S_n\}_0^\infty$ where $S_n = \sum_{k=0}^{n} a_k$, is $(A)$ convergent to $L$.

### 1.1.2 Cesàro convergence

**Definition 1.1.4.** Let $\{a_n\}_0^\infty$ be a sequence of complex numbers.The sequence $\{a_n\}_0^\infty$ in Cesàro convergence(written (C,1) convergent)to $L$
if $\lim_{n\to\infty} \frac{1}{(n+1)} \sum_{k=0}^{n} a_k = L$ exists.

**Definition 1.1.5.** Let $f(x)$ and $g(x)$ be given functions let $x_0$ be a fixed point and suppose that $g(x)$ is positive and continuous in an open interval about $x_0$.
**(i)** If there is a constant $K$ such that $|f(x)| < Kg(x)$ in an open interval about $x_0$ then $f(x) = O(g(x)), (x \to x_0)$
**(ii)** $\lim_{x\to x_o} \frac{f(x)}{g(x)} = 0$ then $f(x) = o(g(x)), (x \to x_0)$
**(iii)** $\lim_{x\to x_o} \frac{f(x)}{g(x)} = 1$ then $f(x) \backsim (g(x)), (x \to x_0)$

**Examples**

(i) $5x^2 + 2x + 3 = O(x^2), (x \to \infty)$, since for $X > 3|5x^2 + 2x + 3| = 5x^2 + 2x + 3 < 6X^2$.
(ii) $sinx = o(x), (x \to \infty)$ since $|sinx| \leqslant 1, \lim_{x\to\infty} \frac{sinx}{x} = 0$.
(iii)$sinx \sim x \quad (x \to 0), \lim_{x\to o} \frac{sinx}{x} = 1$.

### 1.1.3 Summability Method

A summability method $\upsilon$ is a triple $(V, N_v, V - \lim)$ consisting of
(i) A map $V : D_v \to M$,where $D_v \subset W$ and $M$ is a set such that at least on a suitable subset $\mathbb{N}, \phi \neq N \subset M$,there exist a (standard) limit function $f : N \to \mathbb{K}$.
(ii) The domain

$$N_v : V^{-1}(N) \;\; of \;\; \gamma$$

and
(iii) The **summability functional**

$$V - \lim := F_0 V|_{N_v} : N_v \to \mathbb{K}$$

Each $x \in N_v$ is called summation or $\nu$-**summable** to $V - lim\ x$. A series $\sum_{\nu} a_\nu$ is called **summable** (by $\nu$ to the value $\alpha$), if $(x_k)$ with $x_k := \sum_{\nu=0}^{k} a_\nu (k \in N^0)$ is summable (and $\alpha = V - lim\ x$ holds).

**Example 1.1.6.**
Let $M := \omega,\ N := c,\ f := V - \lim$
putting the values $V := id_\omega : \omega \to \omega, x \to x$ we get $c_v = c$ and'$V - \lim = \lim$; therefore $(id_\omega, c, \lim)$ is a summability method which is just the ordinary notion of convergence.

**Example 1.1.7.**
If $Z_{\frac{1}{2}}, cz_{\frac{1}{2}}$ and lim $z_{\frac{1}{2}}$

Where $n : \omega$ and

$$z_{\frac{1}{2}} : w \to w$$

$$x = (x_k) \to (\frac{x_{n-1} + x_n}{2}) \text{ (with } x_{-1} = 0)$$

as well as $cz_{\frac{1}{2}} = \{x \in \omega | Z_{\frac{1}{2}} x \in c \mid = Z_{\frac{1}{2}}^{-1}(c)$

Then we obtain for every $x \in cz_{\frac{1}{2}}$ a limit(generalized) by the setting

$$z_{\frac{1}{2}} - \lim := \lim z_{\frac{1}{2}} := \lim \ o \ z_{\frac{1}{2}} : cz_{\frac{1}{2}} \to K, x \to \lim \ Z_{\frac{1}{2}} x$$

then $(z_{\frac{1}{2}}, cz_{\frac{1}{2}}, \lim \ z_{\frac{1}{2}})$ is summability method which is a called the **Zweier Method**(of order $\frac{1}{2}$).

### 1.1.4 Matrix Method

$$A = (a_{nk}) = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0k} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

with $[A]_{nk} := a_{nk} \in \mathbb{K}(n, k \in \mathbf{N}^*)$ infinite matrix and let $\omega_A := \{x = (x_k) \in \omega | Ax := (\sum_k a_{nk} x_k)$ exists, that is all series $\sum_k a_{nk} x_k$ *converges*$\}$
$A : \omega_A \to \omega, x \to A_x$ (matrix map induced by the matrix $A$) $C_A := \{ x \in \omega | Ax \in C\} = A^{-1}(C)$ **domain of A**)
$\lim_A := A - \lim \ := \lim \ o \ A : C_A \to \mathbb{K}, x \to \lim \ Ax$, then the summability method $(A, C_A, \lim A)$ is called a matrix method.

Moreover,
$C_{0\ A} := \{x \in C_A | \lim_A x = 0\} = A^{-1}(C_0)$ and $m \cap C_A$ are called the **null domain** and the **bounded domain** of $A$, respectively.

**Example 1.1.8.**

$$A := I := \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

is there the identity matrix. Then the corresponding matrix methods yields, obviously the summability method $(id_w, C, \lim)$.

#### 1.1.4.1 Abel Method

let $M = N = C$

let $F : C \to \mathbb{K}, \quad f \to F(f) : \lim\limits_{t \to 1 - t \in \mathbb{R}} f(t)$ Then $(A_1, CA_1, A_1 - \lim)$ is a summability method and is called as Abel Method.

Where

$C := \{f : D_1 \to \mathbb{K} | \lim\limits_{t \to 1 - t \in D_1 \cap \mathbb{R}} f(t) \;\; exists\}$,

$$(D_1 := \{x \in \mathbb{K} ||Z| < 1\})$$

$C_{A_1} := \{x = (x_k) \in \omega | g(t) : \sum\limits_k x_k t^k \;\; converges \;\; for \;\; |t| < 1, and \;\; f \in c \;\; is \; given \; by \;\; f(t) = (1-t)g(t)\}$

and $A_1 : C_{A_1} \to C, x \to f$ with $f(t) = (1-t)g(t)$ then a (generalized limit is defined on $C_{A_1}$ by $A_1 - \lim : C_{A_1} \to \mathbb{K}, x = (x_k) \to \lim\limits_{t \to 1 - t \in \mathbb{R}} \sum\limits_k x_k t^k$.

### 1.1.5 Borel Method

Let $\tilde{C} = \{f : \mathbb{K} \to K | \lim\limits_{t \in \mathbb{R} \to +\infty} e^{-t} f(t) \;\; exists\}$

$$\tilde{C}_{B_1} = \left\{ (x_k) \in \omega | f \in \tilde{C} \;\; \text{where} f(t) = \sum_k \frac{x_k}{k!} t^k \; (t \in \mathbb{K}) \right\},$$

$$B_1 : \tilde{C}_{B_1} \to \tilde{C}, (x_k) \to f \text{with } f : \mathbb{K} \to \mathbb{K}, t \to f(t) := \sum_k \frac{x_k}{k!} t^k$$

and $B_1 - \lim : \tilde{C}_{B_1} \to \mathbb{K},\ (x_k) \to \lim_{t \in \mathbb{R} \to +\infty} e^{-t} \sum_k \frac{x_k}{k!} t^k$ then $(B_1, \tilde{C}B_1, B_1 - \lim)$ is a summability method which is called the **Borel method**.

**Definition 1.1.9.** Rearrangement
if $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ is objective map, then for a given series $\sum_v a_v$ the series $\sum_v a_{\sigma(v)}$ is called a re-arrangement of it.

**Theorem 1.1.10.** ***(Silverman-Toeplitz Theorem)****[4]*
*A matrix $A = (a_{n,k})$ is regular of*

*(i)* $\lim_{n \to \infty} a_{n,k} = 0$ *for each k=0,1,2,3......*

*(ii)* $\lim_{n \to \infty} \sum_{k=0^\infty} a_{n,k} = 1,$ *and*

*(iii)* $\sup_n \{ \sum_{k=0^\infty} |a_{n,k}| \} \leqslant M < \infty$ *for some* $M > 0.$

For $n, m \in \mathbb{N}$ with $n < m$ , let $[n, m]$ denote the set $\{n, n+1, n+2, n+3....n\}$ let $A \subset \mathbb{N}$
Let us define $\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \bigcap [1,n]|}{n}$

and

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{|A \bigcap [1,n]|}{n}$$

The numbers $\overline{d}(A)$ and $\underline{d}(A)$ are called the upper natural density and the lower natural density of $A$, Respectively if $\overline{d}(A) = \underline{d}(A)$ ,then this common value is called the natural density of $A$. and we denote it is by $d(A)$ let $I_d$ be the family of all subsets of $\mathbb{N}$ which have natural density 0.Then $I_d$ is a proper nontrivial admissible deal of subsets of $\mathbb{N}$ (A family $I \subset 2^n$ of subsets on nonempty set N is said to be an ideal in $\mathbb{N}$ .If(i)$A, B \in I \Rightarrow A \cup B \in I$

(ii)$A \in I$, $B \subset A \Rightarrow B \in I$

Further if

$\bigcup_{A \in I}$ =$\mathbb{N}$

Which implies that $\{K\} \in I$ or each $K \in \mathbb{N}$ then $I$ is called admissible or free. I is proper and nontrivial if $\mathbb{N} \notin I$ and$I \neq 0$.

**Definition 1.1.11.** Let $(x_n)$ be a sequence of reals we say that $(x_n)$ tends to $y$ statistically provided $\{n : |x_n - y| \ngeq \in\} \in I_d$ for every $\in > 0$ a sequence $(x_n)$ tends to $y$ in the sense of cesàro if,
$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} n_k = y$.

## 1.2 The Relation Between The Cesàro Summability And The Statistical Convergence

(i) If $x_n \in l^\infty$ is statistically convergent to $y$, then $x_n$ tends to $y$ in the sense of **cesàro**.

(ii) Fridy stated that there is an unbounded sequence $(x_n)$ which is statistically convergent to some $y$ to $x_n$ tends to $\infty$ in the sense of **cesàro** let $a_n = (-n)^n$ if $n$ is even then,

(iii) If $(x_n)$ is a sequence non-negative real number statistically convergent to zero,then $(x_n)$ tends to zero in the sense of cesàro.Converse not hold to see this,consider the following example.
Let $x_n$=a if 3 divides n and put $x_n$=b if 3 does not divide n, $a \neq b$
So $(x_n)$ tends to $\frac{a+2b}{3}$ in the sense of cesàro but not statically convergent as

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = a \times d(\{n : x_n = a\}) + b \times d(\{n : x_n = b\})$$

**Definition 1.2.1.** A function $\delta$, defined for all sets of natural numbers and taking values in the closed interval $[0, 1]$, will be called a lower asymptotic density (or simply density ) if the following four axioms hold:

**(D·1)** If A $\sim$ B, then $\delta(A) = \delta(B)$;

**(D·2)** If A $\bigcap$ B= $\phi$, then $\delta(A) + \delta(B) \leq \delta(A \bigcup B)$;

**(D·3)** For all A, B, $\delta(A) + \delta(B) \leq 1 + \delta(A \bigcap B)$;

**(D·4)** $\delta(I) = I$.

If $\delta$ is any density, we define, $\overline{\delta}$, the upper density associated with $\delta$, by

$$\overline{\delta}(A) = 1 - \delta(IA)$$

for any set of natural numbers A. For a non-negative regular matrix A and $E \subset \mathbb{N}$, we define the $A$- density of $E$ denoted by $\delta_A(E)$, as follows.

$$\begin{aligned}
\overline{\delta_A(E)} &= \lim_{n\to\infty} sup \sum_{k\in E} a_{n,k} \\
&= \lim_{n\to\infty} sup \sum_{k=1}^{\infty} a_{n,k} 1_E(K) \\
&= \lim_{n\to\infty} sup(A1_E)_n
\end{aligned}$$

Similarly

$$\begin{aligned}
\underline{\delta_A(E)} &= \lim_{n\to\infty} inf \sum_{k\in E} a_{n,k} \\
&= \lim_{n\to\infty} inf \sum_{k=1}^{\infty} a_{n,k} 1_E(K) \\
&= \lim_{n\to\infty} inf(A1_E)_n,
\end{aligned}$$

where $1_E$ is a 0-1 sequence such that $1_E(k) = 1 \Leftrightarrow$. if $\overline{\delta_A(E)} = \underline{\delta_A(E)}$, then we say that the $A-$density of $E$ exists and is denoted by $\delta_A(E)$. It is obvious that, if $A$ is the cesàro matrix i.e.,

$$a_{nk} = \begin{cases} \frac{1}{n}; & \text{if n greater than equal to k} \\ 0; & \text{otherwise.} \end{cases}$$

then $\delta$(A) coincides with the natural matrix.

**Lemma 1.2.2.** Let A be a non-negative regular matrix and $E_i = \{V(j)\}$ an infinite subset of $\mathbb{N}$. If $\delta_A(E)$ exists, then $A^{[E]}$ is $\delta_A(E)$ multiplicative. Conversely, if $\delta_A(E)$ is $t$-multiplicative, then $\delta_A(E)$ exists and equals $t$.

*Proof.* Since $A^{[E]}$ is common sub-matrix of A.$A^{[E]}$ satisfies conditions of Silverman,Toeplitz.

For any n,

$(A^{[E]_e})_n = \sum_{k=1}^{\infty} a_{n,r(k)} = \sum_{k\in E} a_{n,k}.$

Thus,if $\delta_A(E)$ exists,then $A^{[E]}$ is $\delta_A(E)$ multiplicative.

Conversely,if $(A^{[E]}$ is $t-multiplicative$,then

$$t = \lim_{n\to\infty} (A^{[E]_e})_n = \lim_{n\to\infty} \sum_{k\in E} a_{n,k} = \delta_A(E)$$

□

**Theorem 1.2.3.** *[3] Assume that A is non-negative regular summability matrix. Assume that* $(x_n) \in l^\infty$ *is a splice over a partition* $\{E_i\}$. *Let* $y_i = \lim_{n\to\infty, n\in E_i} x_n$. *Assume that* $\delta_A(E_i)$ *exists for each i and* $\sum_i \delta_A(E_i)$=*1. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}x_k = \sum_i y_i\delta_A(E_i)$$

.

*Proof.* Assume that A is non-negative regular summability matrix.Assume that $(x_n) \in l^\infty$ is a splice over a partition $\{E_i\}$. Let $y_i = \lim_{n\to\infty,n\in E_i} x_n$. Assume that $\delta_A(E_i)$ exists for each $i$ and $\sum_i \delta_A(E_i)$=1,and let x be a bounded

$\infty - splice$ over $\{E_i\}$.Then for a given n,

$$\begin{aligned}
(Ax)_n &= \sum_{k=1}^{\infty} a_{n,k}\dot{x_k} \\
&= \sum_{i=1}^{\infty}(\sum_{k\in E_i} a_{n,k}\dot{x_k}) \\
&= \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_{nv_i(j)}x_{\dot{v_i(j)}}) \\
&= \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_{nv_i(j)}\gamma_{\dot{j(i)}}) \\
&= \sum_{i=1}^{\infty}((A^{[E_i]}\dot{\gamma^i})_n
\end{aligned}$$

For a fixed n,define
$f_n : N \to C$ and $g_n : N \to C$ by
$f_n(i) := ((A^{[E_i]}\dot{\gamma^i})_n$ and $g_n(i) = M((A^{[E_i]_e})_n$, where $M = ||x||_\infty$ since $\delta_A(E_i)$ exists for each $i$,so by above result,$A^{[E_i]}$ is $\delta_A(E_i) - multiplicative$.Thus,

$$\begin{aligned}
f(i) &:= \lim_{n\to\infty} f_n(i) \\
&:= \lim_{n\to\infty}(A^{[E_i]}\dot{\gamma^i})_n \\
&= \delta_A(E_i)\dot{y_i}
\end{aligned}$$

$$\begin{aligned}
and\ \ g(i) &:= \lim_{n\to\infty} g_n(i) \\
&= \lim_{n\to\infty} M((A^{[E_i]_e})_n \\
&= M\delta_A(E_i).
\end{aligned}$$

If $\mu$ represents counting measure,we have

$$\begin{aligned}
\lim_{n\to\infty}\int_N g_n(i)d\mu &= \lim_{n\to\infty}\sum_{i=1}^{\infty} M((A^{[E_i]_e})_n \\
&= M\lim_{n\to\infty}\sum_{i=1}^{\infty}(\sum_{k\in E_i} a_{n,k}) \\
&= M\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}.
\end{aligned}$$

Since A is regular and $\sum_i \delta_A(E_i)$=1,

$$\begin{aligned}
\lim_{n\to\infty}\int_N g_n(i)d\mu &= M\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k} \\
&= M\cdot 1 \\
&= M\sum_{i=1}^{\infty}\delta_A(E_i) \\
&= \int_N g(i)d\mu
\end{aligned}$$

That is,

$$\lim_{n\to\infty}\int_N g_n(i)d\mu = \int_N \lim_{n\to\infty} g_n(i)d\mu \tag{1.1}$$

Also for every n,

$$|f_n(i)| = |(A^{[E_i]}\dot{\gamma^i})_n| = |\sum_{j=1}^{\infty} a_{nv_i(j)}\dot{\gamma_j^i}| \leqslant \sum_{j=1}^{\infty} a_{nv_i(j)} = M(A_n^{[E_i]_e} = g_n(i) \tag{1.2}$$

so now applying (1.1)and(1.2) on the Lebesgue Dominated Convergence theorem we get,

$$\begin{aligned}
\lim_{n\to\infty}(A\dot{x})_n &= \lim_{n\to\infty}\sum_{i=1}^{\infty}(A^{[E_i]}\dot{\gamma^i})_n \\
&= \sum_{i=1}^{\infty}\lim_{n\to\infty}(A^{[E_i]}\dot{\gamma^i})_n
\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k}\dot{x_k} = \sum_{i=1}^{\infty}\delta_A(E_i)\dot{y_i} = \sum_i y_i\delta_A(E_i).$$

□

By $l^\infty$ we denote the set of all bounded sequences of reals. Fix $(x_n) \in l^\infty$,for $y \in R$ ,let

$$\overline{\delta}_A(y) = lim_{\epsilon\to 0+}\overline{\delta}_A(\{n : |x_n - y| \leqslant \epsilon)$$

and

$$\underline{\delta}_A(y) = lim_{\epsilon\to 0+}\underline{\delta}_A(\{n : |x_n - y| \leqslant \epsilon)$$

If $\overline{\delta}_A(y) = \underline{\delta}_A(y)$,then the common vaue is denoted by $\delta_A(y)$.

## 1.3 A-Limit For Sequences With Positive Densities $\delta_A(Y)$ OR $\overline{\delta}_A(Y)$

**Lemma 1.3.1.** Suppose that $\delta_A(y)$ exists for any $y \in \mathbb{R}$.Then the set $D = \{y \in \mathbb{R} : \delta_A(y) > 0\}$ is countable and $\sum\limits_{y\in D} \delta_A(y) \leq 1$.

*Proof.* Let $x_n$ be a strictly monotonically decreasing sequence converging to 1.For $m \in \mathbb{N}$ let $D_m = \{y \in \mathbb{R} : \delta_A(y) \geq \frac{1}{m}\}$. Let $y_1, y_2, ......, y_l \in D_m$ be distinct.Then for $\epsilon = min_{i\neq j}\dfrac{|y_i - y_j|}{3} > 0$ the sets $E_i = \{n : |x_n - y_i| \leq \epsilon\}$ are pair wise disjoint and $\delta_A(E_i) \geq \dfrac{1}{m}$.

Since A is also regular so we can choose a $n_0$ such that

$\sum\limits_{k\in E_i} a_{n,k} \geq \dfrac{1}{m}$ and $\sum\limits_{k=1}^{\infty} a_{n,k} \leq x_p$ for $n \geq n_0$ and for all $i = 1, 2, ......, l$ where p is fixed.

Since $E_1, E_2, ........, E_l$ are pairwise disjoint ,so

$$\sum_{k\in E_1\cup E_2\cup...\cup E_l} a_{n,k} = \sum_{j=1}^{l}\sum_{k\in E_j} a_{n,k} \geq \frac{l}{m}, \qquad \text{for } n \geq n_0$$

Therefore we must have $l \leq m[x_p]$,where $[x_p]$ denotes the greatest integer function.Hence $D_m$ must be finite and also $\sum_{y\in D_m} \delta_A(y) \leq x_p$.
Since $D_1 \supset D_2 \supset .......$ and $D = \cup_m D_m$ ,we obtain

$\sum\limits_{y\in D} \delta_A(y) = \lim\limits_{m\to\infty} \sum\limits_{y\in D_m} \delta_A(y) \leq x_p$.Since this is true for every $x_p$ and $x_p \to 1$ so we must have

$$\sum_{y\in D} \delta_A(y) \leq 1.$$

Hence D must be countable. □

**Remark 1.3.1.** The above results would not remain true if would change $\delta_A(y)$ to $\overline{\delta_A(y)}$, that is
$\overline{D} := \{y \in \mathbb{R} : \overline{\delta_A(y)} > 0\}$ need not be countable.

**Proposition 1.3.2.** *There is a bounded sequence* $(x_n)$ *such that*
$\overline{\delta_A}(\{y \in \mathbb{R} : |x_n - y| \leq \epsilon\}) = \overline{d}(\{y \in \mathbb{R} : |x_n - y| \leq \epsilon\}) = 1$ *for any* $\epsilon > 0$ *and any* $y \in [0,1]$ *where* $A$ *is the Cesàro matrix.*

*Proof.* Let $(z_n)$ be a sequence such that its set of limit points equals $[0,1]$.Let define $(z_n)$ in such that any rational number from $[0,1]$ appears infinitely many times in the sequence $(z_n)$.

Let $n_k = 10^{k^2}$,then

$$\begin{aligned}
\frac{|[n_k + 1, n_{k+1}]|}{n_{k+1}} &= \frac{10^{(k+1)^2} - (10^{k^2} + 1)}{10^{(k+1)^2}} \\
&= \frac{10^{k^2+k+1} - 10^{k^2} - 1}{10^{(k+1)^2}} \\
&= \frac{10^{k^2} \cdot 10^{2k} \cdot 10}{10^{k^2} \cdot 10^{2k} \cdot 10} - \frac{10^{k^2}}{10^{k^2} \cdot 10^{2k} \cdot 10} - \frac{1}{10^{(k+1)^2}} \\
&= 1 - \frac{1}{10^{2k+1}} - \frac{1}{10^{(k+1)^2}} \\
&\to 1.
\end{aligned}$$

Let $B_0 = [0, n_1]$ *and* $B_k = [n_k + 1, n_{k+1}]$ *for* $k \geq 1$.If A consists of infinitely many $B_k^\prime s$, then $\overline{d}(A) = 1$. Let $x_n = z_k$ if $n \in B_k$. Let $y \in [0,1]$. Then for every $\epsilon > 0^p$ the set
$C := \{k : |z_k - y| < \epsilon\}$ is infinite.
$A := \{n : |x_n - y| < \epsilon\} = \cup_{k \in C} B_k$.
Therefore
$\overline{d}(A) = 1$.

□

**Theorem 1.3.3.** *[1] suppose that* $x = (x_n)$ *is a bounded sequence* $\delta_A(y)$ *exists* *for every* $y \in \mathbb{R}$ *and* $\sum_{y \in D} \delta_A(y) = 1$. *Then*

$$\lim_{n \to \infty} (ax)_n = \sum_{y \in D} \delta_A(y) \cdot y$$

.

*Proof.* Since $(x_n)$ is a bounded sequence there exist a +ve number $M$ that is $M > 0$ such that $|(x_n)| < M$ for every $n \in N$.Let D $=\{y_1\}_i$ that is $y_1, S$ are distinct. Let $n \in > 0$ be given and let $r \in \mathbb{N}$ be such that

$$\sum_{i=1}^{r} \delta_A(y_i) > 1 - \epsilon$$

and

$$\sum_{i=r+1}^{\infty} \delta_A(y_i), y_i < \epsilon$$

Let

$$N \in \mathbb{N}$$

be such that

$$\frac{1}{3} \min_{1 \leqslant i \neq j \leqslant r} |y_i - y_j| > \frac{1}{N}$$

and which that the set

$$E_i := \{j : |x_j - y_i| < \frac{1}{n}$$

have the following property

$$\delta_A(y_i) \leqslant \delta_A(E_i) \leqslant \overline{\delta}_A(E_i) \leqslant \delta_A(y_i) + \frac{\epsilon}{rM_o}$$

for i = i,......,r where $M_0 = \max\{|y_1|, |y_2|, ....|y_r|\}$

where$E_1, E_2, ......E_r$ Are pare wise disjoint now choose $a$ $M_0 \in \mathbb{N}$ such that

$$\delta_A(E_i) - \frac{1}{N} < \sum_{k\in E_i} a_{n,k} < \delta_A(E_i) + \frac{1}{n}$$

$$for every n \geqslant n_0$$

and

$$i = 1, .........r$$

therefore,

$$\delta_A(y_i) - \frac{1}{n} - \frac{\epsilon}{rm_0} < \sum_{k\in E_i} a_{n,k} < \delta_A(y_i) + \frac{1}{n} + \frac{\epsilon}{rM_o}$$

for every $n \geqslant m_0$ And $i = 1, 2, .......r$.Then for $n \geqslant m_0$ we have

$(Ax_n) = \sum_{k=1}^{\infty} a_{n,k}x_k \leqslant \sum_{k\in E_1} a_{n,k}(y_1 + \frac{1}{n} + ........... + \sum_{k\in E_r} a_{n,k}(y_r \frac{1}{n})$
since $A$ is regular, we can choose $m_1 \geqslant m_0$ such that for all $n \geqslant m_1$
$\sum_{k=1}^{\infty} a_{n,k} < 1 + \epsilon$

now the observe that

$$1 + \epsilon > \sum_{k=1}^{\infty} a_{n,k} = \sum_{k\in(E_1\cup...E_r)} a_{n,k} + \sum_{k\in(E_1+\cup....\cup E_r)}^{ca_{n,k}}$$

where from above we have

$$\sum_{k\in E_1\cup...\cup E_r} a_{n,k} = \sum_{j=1}^{r}\sum_{k\in E_j} a_{n,k} > \sum_{j=1}^{r} \delta_A(y_i) - \frac{r}{N} - \frac{\epsilon}{M_0} > 1 - \frac{r}{N} - (1 + \frac{1}{M_o}).\epsilon$$

therefore for $n \geq m_o$ we have

$$(Ax_n) \leqslant_{k\in E_1} a_{n,k}(y_1 + \frac{1}{N}) + ..... + \sum_{k\in E_r} a_{n,k}(y_r + \frac{1}{N}) + \frac{Mr}{N} + (2 + \frac{1}{M_0})M\epsilon$$

and

$$(Ax_n) \geq_{k\in E_1} a_{n,k}(y_1 - \frac{1}{N}) + ..... + \sum_{k\in E_r} a_{n,k}(y_r - \frac{1}{N}) - \frac{Mr}{N} - (2 + \frac{1}{M_0})M\epsilon$$

Hence for $\geq m_0$

$$\mid (Ax)_n - \sum_i \delta_A(y_i).y_i \mid\leqslant |(Ax)_n - \sum_{i=1}^{r} \delta_A(y_i).y_i \mid +\epsilon$$

$$\leqslant \sum_{i=1}^{r} \mid \sum_{k\in E_i} a_{n,k}.(y_i \pm \frac{1}{N}) - \delta_A(y_i).y_i \mid + \frac{Mr}{N} + (2M + \frac{M}{M_0+1})\epsilon$$

$$\leqslant \sum_{i=1}^{r} \mid (\sum_{k\in E_i} a_{n,k} - \delta_A(y_i)(y_i \pm \frac{1}{N}) + \frac{Mr}{N} + (2m + \frac{M}{M_0} + 1)\epsilon$$

$$\leqslant (\frac{1}{N} + \frac{\epsilon}{rM_0}).r.(M_0 + \frac{1}{N}) + \frac{r}{N} + \frac{Mr}{N} + (2m + \frac{M}{M_0} + 1)\epsilon$$

since $N$ can be choosen arbitarily large,we obtain

$$\mid (Ax)_n - \sum_i \delta_A(y_i).y_i \mid\leqslant (2M + \frac{M}{M_0} + 2)$$

for every $\epsilon > 0$.Therefore

$$\lim_{n\to\infty} (Ax)_n = \sum_i \delta_A(y_i) \cdot\ y_i$$

. □

**Theorem 1.3.4.** *[1] Assume that $x = x_n$ is bounded if $\overline{\delta}_A(y) = 1$. Then $y$ is a limit point of the sequence $((Ax)_n)$.*

*Proof.* Since $(x_n)$ is bounded there exist $M > 0$ such that $|x_n| \leqslant M$ for every $n \in \mathbb{N}$. Let $y \in \mathbb{R}$ such that $\overline{\delta}_A(y) = 1$ let $n \in \mathbb{N}$ let $E_N = j \in \mathbb{N} : |x_j - y| < \frac{1}{p}$ where

$P = P(N) > N$ is such that $1 \leqslant \overline{\delta}_A(E_N) \leqslant 1 + \frac{1}{2N}$ Then there is $K_n \geq \mathbb{N}$ such that

$$\sum_{k \in E_N} a_{kn,k} > \overline{\delta}_A(E_N) - \frac{1}{2N} \geq 1 - \frac{1}{2N}$$

and also from regularity of $A$.

$$\sum_{k=1}^{\infty} a_{kn,k} < 1 + \frac{1}{N}$$

Then we have

$$\sum_{k \in E_N} a_{kn,k} - \sum_{k \notin E_N} a_{kn,k}.M$$

$$\sum_{k=1}^{\infty} a_{kn,k} x_k \leqslant \sum_{k \in E_N} a_{kn,k}.(y + \frac{1}{N}) + \sum_{k \notin E_N} a_{kn,k}.M$$

Hence

$$| (Ax)_{KN} - y | = | \sum_{k=1}^{\infty} a_{kn,k} - y|$$

$$\leqslant (\frac{1}{N} + \frac{1}{N^2}) + \sum_{K \notin E_N} a_{kn,k}(m + |y|) + \frac{|y|}{N}$$

$$\leqslant (\frac{1}{2N} + \frac{1}{N})(M + | y |) + \frac{| y | + 1}{N} + \frac{1}{N^2}$$

therefore

$$\lim_{n \to \infty} (Ax)_{KN} = y$$

that $y$ is a limits points of the sequences$((Ax_n))$. □

From the above theorem we obtain the following corollary.

**Corollary 1.3.5.** *let $(x_n)$ be a bounded sequence. Suppose that there are $y$ and $z$ $(y \neq z)$.with $\overline{\delta}_A(y) = \overline{\delta}_A(z) = 1$. Then the $A$ limit does not exit.*

*Proof.* we prove it by method of contradiction
Since $(x_n$ be a bounded sequence,and $\bar{\delta_A}(y) = 1$ so by the above theorem(1.3.4).we get $y$ is a limit point of $x_n$ that is

$$\lim_{n\to\infty} (Ax)_{KN} = y \tag{1.3}$$

and also $\bar{\delta_A}(z) = 1$, so $z$ is a limit point that is

$$\lim_{n\to\infty} (Ax)_{KN} = Z \tag{1.4}$$

from equation(1.3) and (1.4) we get,

$$\lim_{n\to\infty} (Ax)_{KN} = y = z \tag{1.5}$$

which is a contradiction as given that$y \neq z$ so the $A$ limit

$$E \lim_{n\to\infty} (Ax)_N$$

does not exist. □

**Proposition 1.3.6.** *Let $t \in \mathbb{N}, r, s \in [1, 2^t - 1]$ and $L \in \mathbb{R}$ with $y \neq Z$ then there is a sequence $(x_n)$,such that*
$\overline{d}(y) = \frac{r}{2^t}$,

$\overline{d}(y) = \frac{s}{2^t}$ *(That is when we are taking the limit with respect to cesàro matrix) and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = L$$

.

*Proof.* let $y^{'}$ and $z^{'}$ be such that $\frac{yr}{2^t}+y^{'}\frac{(1-r)}{2^t}=\frac{zs}{2^t}+z^{'}\frac{(1-s)}{2^t}=L$. Define $n_k$ as follows. Let $n_0=0$ and $n_k=2^t10^{k^2}, k\in\mathbb{N}$ let

$$B_k=[n_{k-1}+1,n_k], \qquad k\in\mathbb{N}$$

$$\mid B_k\mid=2^t(10^{k^2}-10^{(k-1)^2})$$

let $A_k \subset B_k$ be defined as follows.

$$A_{2k}=\bigcup_{n=1}^{10^{(2k)^2}-10^{(2k-1)^2}} [n_{2k-1}+1+m2^t, n_{2k-1}+1+m2^t+r]$$

$$A_{2k+1}=\bigcup_{n=1}^{10^{(2k+1)^2}-10^{(k)^2}} [n_{2k}+1+m2^t, n_{2k}+1+m2^t+s]$$

now we define $x_n$, let $x_n=y$ in $n\in A_{2k}$

$x_n=y^{'}\ if\ n\in\frac{B_{2k}}{A_{2k}}$, $x_n=z$ if $n\in A_{2k+1}$ and $x_n=z'$ if $n\in\frac{B_{2k+1}}{A_{2k+1}}$

$$\bar{d}(y)=\bar{d}(\bigcup_{k=1}^{\infty}A_{2k})=\frac{r}{2^t}$$

,

$$\bar{d}(y')=\bar{d}(\bigcup_{k=1}^{\infty}\frac{B_{2k}}{A_{2k}})=1-\frac{r}{2^t}$$

,

$$\bar{d}(z)=\bar{d}(\bigcup_{k=0}^{\infty}A_{2k+1})=\frac{s}{2^t}$$

and

$$\bar{d}(z')=\bar{d}(\bigcup_{k=0}^{\infty}\frac{B_{2k+1}}{A_{2k+1}})=1-\frac{s}{2^t}$$

so for any $k \in \mathbb{N}$ and $m = 1, ...., 10^{(2k)^2} - 10^{(2k-1)^2}$ we have

$$\sum_{i=n_{2k-1}+1+m2^t}^{n_{2k-1}+1+(m+1)_{2^t}} x_{i=} = ry + (1-r)y^1 = 2^t L$$

similarly, for any $K \in \mathbb{N}$ and

$m = 1........10^{2k+1^2} - 10^{2k^2}$ We have

$$\sum_{i=n_{2k}+1+m2^t}^{n_{2k+1}+1+(m+1)\cdot 2^t} x_{i=} = sz + (1-s)z^{'} = 2^t L$$

from this we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = L.$$

□

## 1.4 Relation Between A-Statistical Limit Points And Points Having Positive A-Density

Let $I$ be a proper non-trivial admissible ideal in $\mathbb{N}$ And let $(x_n) \in l^{\infty}$.We say that a sequence $(x_n)$ of real numbers tends to $y$ with respect to $I$ provided

$\{n :| x_n - y |\geq \epsilon\} \in I$ for every $\epsilon > 0$, in symbols $y = I - \lim_{n\to\infty} x_n$.

**Proposition 1.4.1.** *[4] If $I$ is a maximal ideal and $(x_n) \in l^\infty$ then $I - \lim_{n\to\infty} x_n$ exists.*

*Proof.* suppose that $I$ is a maximal admissible ideal in $\mathbb{N}$. Let $x = (x_n) \in l^\infty$ we show that there exists $\xi \in \mathbb{R}$ such that $I - \lim_{n\to\infty} x_n \in \mathbb{R}$. Since $x \in l^\infty$ there are numbers $a, b \in \mathbb{R}$ such that $a \leqslant x_n \leqslant b, (n = 1, 2, 3.......)$.Put $A_1 = n : a \leqslant x_n \leqslant \frac{a+b}{2}$, $B_1 = n : \frac{a+b}{2} \leqslant x_n \leqslant b$ then $A_1 \cup B_1 = \mathbb{N}$. Since $I$ is admissible ideal both lets $A_1, B_1$ cannot belongs to $I$ thus atleast one of them does not belongs to $I$.Denote it by $D_1$ and interval corresponding to it done by $I_1$. so we have(infinite) set $D_1 = n : x_n \in I_1 \notin I_1$
so we can (by introduction) construct a sequence of closed intervals $I_1 \geq I_2 \geq ............, I_n = [a_n, b_n]$ $(n = 1, 2, 3......, lim_{n\to\infty}(a_n - b_n) = 0$ and sets $D_k = n : x_n \in l_k \notin I(k = 1, 2, 3....)$

let

$$\xi \in \cap_{k=1}^{\infty} I^k$$

and $\epsilon > 0$ construct the set

$$M(\epsilon) = \{n : |x_n - \xi| < \epsilon\}$$

sufficiently large $m$ we have
$I_m = [am, bm] \leqslant (\xi - \epsilon, \xi + \epsilon)$. Since $D_m \notin I$ we use that $M(\epsilon) \notin I$

Since $M(\epsilon) \notin I$, The maximality of $I$ implies that

$A(\epsilon) = \{n : |x_n - \xi| \geq \epsilon\} = \frac{N}{M}(\epsilon) \in I$

Hence

$$I - \lim_{n\to\infty} x_n = \xi$$

□

*A point $y$ is called $I$-cluster point of $(x_n)$ if $\{n : |x_n - y| \leqslant \epsilon\} \notin i$. We say $y$ an $I$-limit point$(x_0)$ if there is a set $B \subset \mathbb{N}, B \notin I$ such that $\lim_{n\in B} x_n = y$. Since $I$ contains all singletons,clearly $I$-limit points are $I$ cluster points and $I_d$-limit points are called statistical cluster points and statistical limit points.Respectively whole $I_A$-cluster points and $I_A$-limit points are called $A$-statistical cluster points and $A$ statistical limit points respectively. where $I_A = (B \subset \mathbb{N} : \delta_A B = 0)$ forms an admissible ideal in $\mathbb{N}$*

**Lemma 1.4.2.** Let $I$ be an ideal of subsets of $\mathbb{N}$.Assume that $X := \{n : x_n \in (a, b]\} \notin I$ suppose that $\{n : a \leqslant x_n \leqslant t - \epsilon\} \in I$
or $\{n : t + \epsilon \leqslant x_n \leqslant b\} \in I$. for any $t \in (a, b$ and any $\epsilon > 0$ such that $\epsilon < min\{t - a, b - t\}$, then there is $y \in [a, b]$ such that $n : |x_n - y| \geq \epsilon\} \in I$ for every $\epsilon > 0$.

*Proof.* suppose that for any $y \in [a, b]$ there is $\epsilon_y > 0$ with

$$\{n : |x_n - y| \geq \epsilon_y\} \notin I$$

Since $[a, b]$ is compact there are $a \leqslant y_1 < y_2 < ........ < y_k$ such that $\{(y_{i-\xi}, y_i + \xi_i) : i = 1, 2, 3......., k\}$

is an open cover of $[a, b]$,
where $\xi_i = \xi y_i$ we may assume that none element of this cover contains over element of this sub-cover let $A_i = \{n : |x_n - y_i| < \xi_i\}$

Note that $A_1 \cup ......... \cup A_k = X$ and therefore there is $i$ with $A_i \notin I$. since $X|A_i \in I|$

there is $j \neq i$ such that $\dfrac{A_j}{A_i} \notin I$
assume that $i < j$. Since $y_i < y_j$ and $(y_j - \epsilon_j, (y_j + \epsilon_j)$is not contained in

$(y_i - \epsilon_i, (y_i + \epsilon_i)$.Then $y_i + \epsilon_i < y_j + \epsilon_j$

let $t = y_i + \epsilon_i$ there is $\epsilon_t > 0$ with $B = \{n : |x_n - t| \geq \epsilon_t\} \notin I$. Consider two cases

**Case-I**

If there is $\epsilon > 0$ such that $B' := \{n : |n_n - t| < \epsilon\} \in I$
then $A_i \setminus B\prime \notin I$ and $A_i \setminus (A_2 \cup B') \notin I$

hence $\{n : a \leqslant x_n \leqslant t - \epsilon_t\} \notin I$ and

$$\{n : t + \epsilon_t \leqslant x_n \leqslant b\} \notin I$$

**Case-II**

if $\{n : |x_n - t| < \epsilon\} \notin I$
for any $\epsilon > 0$ then
$\{n : |x_n - t| < \frac{\epsilon t}{2}\} \notin I$
since $B \notin I$ we have either $\{n : G \leqslant x_n \leqslant t - \epsilon_t\} \notin I$
or $\{n : t + \epsilon_t \leqslant x_n \leqslant b\} \notin I$ assume that
$\{n : G \leqslant x_n \leqslant t - \epsilon_t\} \notin I$
Then

$$\{n : G \leqslant x_n \leqslant (t - \frac{3}{4}\epsilon t) - \frac{1}{4}\epsilon_t\} \notin I$$

and

$$\{n : ((t - \frac{3}{4}\epsilon t) + \frac{1}{4}\epsilon_t \leqslant x_n \leqslant b\} \notin I. \qquad \square$$

For any nonempty set $A$, we will denote by $A^{<N}$ family of all finite sequences of elements of $A$. For any finite sequences $s = (s_1 + s_2 .......... + s_n) \in A^{<N}$ and $a \in A$ by $s^\wedge a$

we denote concatenation of $S$ and $a$,That is
$s^\wedge a = (s_1 + s_2 .......... + s_n, a)$ by $\mid S \mid$ we denote the length of $S$. If $\alpha \in A^{\mathbb{N}}$

then let $\frac{\alpha}{n} = (\alpha(1).........., \alpha(n))$ and $\frac{\alpha}{o} = \Phi$ where $\phi$ stands for empty sequence.

An ideal $I$ of $\mathbb{N}$ is called a $P$-ideal if for any sequences of sets $(D_n)$ from $I$ there is another sequences of sets $(C_n)$ in $I$ such that $D_n \Delta C_n$ is finite for every $n$ and

$$\bigcup_n C_n \in I$$

Equivalently if for each sequence $(A_n)$ of sets from $I$ there exists $A_\infty \in I$ such that

$$\frac{A_n}{A_\infty}$$

is finite for all $n \in \mathbb{N}$

A function

$$\phi : 2^{\mathbb{N}} \to [0, \infty]$$

is called a sub-measure if

$$\phi(E) \leqslant \phi(E \cup F)$$
$$\leqslant \phi(E) + \phi(F)$$

for any $E, F \in 2^N A$ sub-measure $\phi$ is called lower semi continuous if

$$\lim_{m \to \infty} \phi(E_n[1, m]) = \phi(E)$$

By $E_{xh}(\phi)$ denote the set of all $E \subset \mathbb{N}$ with

$$\lim_{m \to \infty} \phi(E \diagdown [1, m]) = 0.$$

**Proposition 1.4.3.** *Let $A$ be a non-negative regular matrix.Then $I_A$ is a $P$ -ideal.*

*Proof.* Let

$$\varphi_A(E) = \sup_{n\in\mathbb{N}} \sum_{k\in E} a_{n,k}$$

$$Q\varphi_A(E_1 + E_2) = \sup_{n\in\mathbb{N}} \sum_{k\in E_i+E_2} a_{n,k}$$

$$\leqslant \sup_{n\in\mathbb{N}} \sum_{k\in E_1} a_{n,k} + \sup_{n\in\mathbb{N}} \sum_{k\in E_2} a_{n,k}$$

so$\phi_A$ is monotonous and sub addition now we will show that it is lower semi continuous fix $E \subset \mathbb{N}$ .Let

$$s : \phi_A(E) = \sup_{n\in\mathbb{N}} \sum_{k\in E} a_{n,k}$$

and let $\epsilon > 0$

we can find $n \in \mathbb{N}$ such that

$$\sum_{k\in E} a_{n,k} > s - \frac{\epsilon}{2}$$

.

Then there is $m \in \mathbb{N}$ with

$$s - \epsilon = s - \frac{\epsilon}{2} - \frac{\epsilon}{2} < \sum_{k\in\epsilon, k\leqslant m} a_{n,k} \leqslant \sup_{n\in\mathbb{N}} \sum_{k\in E, k\leqslant m} a_{n,k}$$

$$= \phi_A(E \cap [1, m])$$

Since the sequence $(\phi_A(E \cap [1, m]))$ is non-decreasing, then

$$\lim_{m \to \infty} \phi_A(E \cap [1, m]) = \phi_A(E)$$

Assume that $\delta_A(E) = 0$. Let $\epsilon > 0$ there is $n_0$ such that

$$\sum_{k \in E} a_{n,k} < \epsilon$$

for $n \geq n_o$.Since $\sum_{k \in E} a_{n,k}$ convergent for any $n \geq n_0$,there is $m$ such that

$$\sum_{k \in E, k > m} a_{n,k} < \epsilon$$

for

$$n < n_0$$

consequently,

$$\phi_A(E \diagdown [1, m]) = \sup_{n \in \mathbb{N}} \sum_{k \in E, k > m} a_{n,k} < \epsilon$$

Hence

$$I_A \subset E_{xh}(\phi_A)$$

.

Assume now that

$$\lim_{m \to \infty} \varphi_A(E \diagdown (1, m)) = 0$$

Let $\epsilon > 0$,there is $m \in \mathbb{N}$ such that

$$\sum_{k\in E, k>m} a_{n,k} < \frac{\epsilon}{2}$$

for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} a_{n,k} = 0$ for every $n \in \mathbb{N}$, there is $n_0$ such that $a_{n,k} < \frac{\epsilon}{2m}$ for $n \geq n_0$ and $k \leqslant m$. Therefore $\sum_{k\in E} a_{n,k} < \epsilon$ for $n \geq n_0$. That means that $\delta_A(E) = 0$ , thus $I_A > E_{xh}(\phi_A)$. □

**Proposition 1.4.4.** *Let $I$ be a $P$- ideal.Assume that $(x_n) \in e^{\infty}$ does not have any $I$-limit points .Then the set of limit points of $(x_n)$ that is the set $\{y \in \mathbb{R} : x_{n,k} \to y\ for\ some\ increasing\ sequence\ (n_k)\ of natural\ numbrers\}$ is uncountable and closed.*

*Proof.* Let $I_\phi = [a, b]$ be such that $(x_n) \subset [a, b]$. Then those are $t \in (a, b)$ and $\epsilon > 0$ such that $\{n : a \leqslant x_n \leqslant t - \epsilon\} \notin I$ and $\{n : t + \epsilon \leqslant x_n \leqslant b\} \notin I$ If there are not such $t$ and $\epsilon$. then bye Lemma 1.4.2 there is $y \in [a, b]$ such that $\{n : |x_n - y| \geq \epsilon\} \in I$ for every $\epsilon > 0$ that means that$(x_n)$ is convergent to $y$. Since $I$ is a $P$-ideal so $y$ is an $I$-limit point of $(x_n)$ which yields a contradiction. let $I_{(0)} = [a, t - \epsilon]$ and $I_1 = [t + \epsilon, b]$ proceeding inductively we define a family $\{I_s : s \in \mathbb{N}^{<\mathbb{N}}$ or non trivial compact intervals such that

**(i)**-$I_{s^\wedge i} \subset I_S$ for i=0,1 and $S \in \mathbb{N}^{<\mathbb{N}}$,

**(ii)**-

$$\{n : x_n \in I_s\} \notin I$$

for $\in \mathbb{N}^{<\mathbb{N}}$ so that for any $\alpha \in 0, 1^{\mathbb{N}}$ and any $k \in \mathbb{N}$ there are infinitely many $x_n$ is with $x_n \in I_{\alpha|K}$ Which is a limit point of $(x_n)$. Note that $x_\alpha \neq x_\beta$ for distinct $\alpha, \beta \in 0, 1^{\mathbb{N}}$. Therefore the set of limit points of $(x_n)$ is uncountable and the set of limit points of $(x_n)$ is always closed. □

**Lemma 1.4.5.** let $r \in (0, 1), r, \geq r_2 \geq r_3 \geq .....$

$$\lim_{n\to r} r_n = r$$

and let $(E_n)$ be a decreasing sequences of subsets of $\mathbb{N}$

**(i)**-if $\underline{\delta_A}(E_n) = r_n, n \in \mathbb{N}$, then there is a subset $E$ of $\mathbb{N}$ with $\delta_A(E) = r$ and such that $E \subset^* E_n, n \in \mathbb{N}$

**(ii)** If $\bar{\delta_A}(E_n) = r_n, n \in \mathbb{N}$. then there is a subset $E$ of $\mathbb{N}$ with $\bar{\delta_A}(E) = r$ and such that $E \subset^* E_n, n \in \mathbb{N}$

*Proof.* **(i)** Let $(p_n)$ be a increasing sequence of natural numbers such that $\sum_{k \in E_n} a_{j,k} > r_n - \dfrac{1}{3n}$ for every $j \geq p_n$. For each $n \in \mathbb{N}$ now choose $m_n > p_n$ such that

$$\sum_{k \in E_n \cap [1, m_n]} a_{j,k} > r_n - \frac{1}{3n} - \frac{1}{3n} > r_n - \frac{1}{n}$$

for all

$$j, p_n \leqslant j \leqslant p_{n+1}.$$

Thus we have two increasing sequence of natural numbers $(p_n)$ and $m_n$ such that $\forall j \in [p_n + p_{n+1}]$ we have

$$\sum_{k \in E_n \cap [1, m_n]} a_{j,k} > r_n - \frac{1}{n}$$

put

$$E = \bigcup_{n=1}^{\infty} E_n \cap [1, m_{n+1}]$$

Take $p_n \leqslant j < p_{n+1}$, then

$$\sum_{k \in E} a_{j,k} \geq \sum_{k \in E_n} \cap [1, m_n] a_{j,k} > r_n - \frac{1}{n}.$$

Thus

$$\liminf_{n\to\infty}\sum_{k\in E} a_{n,k} \geq r$$

which means that $\underline{\delta_A}(E) \geq r$
since

$$E_1 \supset \; E_2 \supset ..........$$

so

$$\cup_{n=j}^{\infty} E_n \cap [1, m_{n+1}] \subset E_j$$

and

$$E_j \diagdown E \subset \cup n = 1^{j-1} E_n \cap [1, m_{n+1}]$$

Therefore $E \subset^* E_j$ and consequently $\underline{\delta_A}(E) \leqslant \underline{\delta_A}(E_j)$ and $\bar{\delta_A}(E) \leqslant \bar{\delta_A}(E_j)$.Hence $\underline{\delta_A}(E) = r$ and if $\bar{\delta_A}(E_n) \to r$,then $\delta_A(E) = r$.

**(ii)** As in (i) we choose two increasing sequences of natural numbers $(p_n)$ and $(m_n)$ such that

$$\sum_{k\in E_n\cap[1,m_n]} a_{p_{n,k}} > r_n - \frac{1}{n}$$

for every $n$.
Put

$$E = \bigcup_{n=1}^{\infty} E_n \bigcap \; [1, m_{n+1}]$$

then

$$\sum_{k\in E} a_{p_{n,k}} \geq \sum_{k\in E_n\cap[1,m_n]} a_{p_{n,k}} > r_{n+1} - \frac{1}{n+1}$$

thus $\bar{\delta_A}(E) = r$ and $E \subset^* E_n \qquad \forall n \in \mathbb{N}$. □

**Theorem 1.4.6.** *Let $(x_n) \in l^\infty$. A point $y \in \mathbb{R}$ is an A-statistical limit point of$(x_n)$ if and only if $\bar{\delta_A}(y) > 0$.*

*Proof.* Assume that $\bar{\delta_A}(y) = 0$ and suppose $y$ is an $A$-statistical limit point of $(x_n)$. Then there is $E \subset \mathbb{N}$ such that $\bar{\delta_A}(y) > 0$ and

$\lim_{n\in E} x_n = y$. We have $E \subset^* \{j : |x_j - y| \leqslant \epsilon\}$ or every $\epsilon > 0$.hence $\bar{\delta_A}(E) \leqslant \bar{\delta_A}(E) \leqslant \bar{\delta_A}(\{j : |x_j - y| \leqslant \epsilon\})$ for every $\epsilon > 0$.Therefore $\bar{\delta_A}(E) = 0$ which yields a contradiction. Hence $\bar{\delta_A}(y) > 0$
conversely, assume that $\bar{\delta_A}(y) > 0$. Let $E_n = \{j : |x_j - y| \leqslant \frac{1}{n}$. Then $(E_n)$ is decreasing sequence with $\bar{\delta_A}(E_n) \to \bar{\delta_A}(y)$ By the above lemma (1.4.5) there is $E \subset^* E_n, n \in \mathbb{N}$ with $\bar{\delta_A}(E) = \bar{\delta_A}(y)$ since almost all elements of $E$ are contained in $E_n$. Then clearly

$$\lim_{j\to\infty, j\in\mathbb{E}} x_j = y.$$

Hence $y$ is an $A$-statistical limit points of $(m_n)$. □

# Chapter 2

## *SPLICED SEQUENCES*

**Definition 2.0.1.** Let N be a fixed natural number an N-partition of N consists of N infinite sets $E_1 := \{V_1(j)\}_{j=1}^{\infty}$, $E_2 := \{V_2(j)\}_{j=1}^{\infty}, \cdots\cdots\cdots E_N := \{V_N(j)\}_{j=1}^{\infty}$ such that$N = \bigcup_{i=1}^{N} E_i$ and $E_i \bigcap E_k = \phi$ for $i \neq k$, where $\{V(j)\}$ is an infinite subset of $\mathbb{N}$.

**Definition 2.0.2.** Let $E_1$,$E_2$,$E_3$,$E_4$,$\cdots\cdots\cdots$ $E_k$, be a partion of $\mathbb{N}$ into k sequences and suppose $y_1$,$y_2$,$y_3$ $\cdots\cdots\cdots$,$y_k$ be distinct numbers. Let $(x_n)$ such that

$$\lim_{n\to\infty, n\in E_i} x_n = y_i,$$

Then $(x_n)$ is called a $k-$splice.

**Example 2.0.3.**
Consider the threee partions

$$E_1 = \{1, 3, 5, 7, 9, 11, \cdots\cdots\cdots\}$$
$$E_2 = \{2, 6, 10, 14, 18, 22, \cdots\cdots\cdots\}$$
$$E_3 = \{4, 8, 12, 16, 20, 24, \cdots\cdots\cdots\}$$

and the convergent sequence $a := (a_j)$, $b := (b_j)$ $c := (c_j)$. Then the three splice of the sequences $a, b, c$ over the three partition $\{E_1, E_2, E_3\}$, is the sequence

$$x := \{a_1, b_1, a_2, c_1, a_3, b_2, a_4, c_2, a_5, b_3 \cdots\cdots\}$$

**Definition 2.0.4.** Let A be a regular matrix and consider a fixed $N-$partition $\{E_1, E_2, \cdots\cdots E_N\}$. Then A is said to have the slicing property over $\{E_1, E_2, \cdots\cdots E_N\}$ provided that A sums every $N-$splice over the $N-$ partition $\{E_1, E_2, \cdots\cdots E_N\}$.

**Definition 2.0.5.** An infinite partition of $N$ consists of an infinite number of infinite sets $E_i = \{V_i(j)\}_{j=1}^{\infty}$, $\quad i \in \mathbb{N}$, such that $N = \bigcup_{i=1}^{\infty} E_i$ and $E_i \bigcap E_k = \phi$ for $i \neq k$.

**Proposition 2.0.6.** *Let* $(x_n) \in l^\infty$ *Assume that there exist a distinct real numbers* $y_1, ......, y_m$ *with* $\delta_A(y_i) > 0 \forall \;\; i$ *such that*

$$\sum_{i=1}^{m} \delta_A(y_i) = 1$$

*. Then there exist a partition* $E_1, .......E_n, E_{n+1}$ *such that* $\delta_A E_i = \delta_A(y_i)$*, For* $i = 1, 2, 3...m$
$\delta_A(E_{m+1}) = 0$ *and* $\lim_{n \in E_i} x_n = y_i$.

*Proof.* By theorem 1.4.6 there are$E_{1'}, .......E_{m'}$ with $\lim_{n \in E_i} x_n = y_i$,Let $\epsilon = \min\{|y_i - y_j| \diagup 3 : i, j = 1......m, i \neq j\}$
put

$$E_i = \{n : |x_n - y_i| \leqslant \epsilon\} n E_{'}, i = 1, 2, ......m$$

clearly $\delta_A E_i = \delta_A(y_i)$

$$E_{n+1} = \mathbb{N} \diagdown \bigcup_{i=1}^{n} E_i$$

has A-density zero, $E_1, .......E_n, E_{n+1}$ are pairwise disjoint, and $\lim_{n \in E_I} x_n = y_i$. □

**Proposition 2.0.7.** *Let* $(x_n) \in l^\infty$*. Assume that there exist distinct real numbers* $y_1, y_2, .........$ *such that* $\delta_A(y_i) > 0, \forall i$ *such that*

$$\sum_{i=1}^{\infty} \delta_A(y_i) = 1.$$

*Then there exist a partition* $E_1, E_2, \cdots\cdots E_n, E_{n+1}$ *such that* $\delta_A E_i = \delta_A E_i$*,* $i = 1, 2, .......$ *and* $\lim_{n \in E_i} x_n = y_i$.

*Proof.* By theorem 1.4.6 there are$E_1{}', .......E_m{}'$ with $\lim_{n \in E_i} x_n = y_i$ $E_i{}' \cap E_j^{'}$ is finite if $i \neq j$. Define $E_1, E_2, ...........$in the following way.Let $E_1{}'' = E_1{}'$,

$$E_m{}'' = E_m{}' \diagdown \bigcup_{i=1}^{n-1} E_i$$

$,k \geq z$ since

$$E_m{}' \cap \bigcup_{i=1}^{n-1} E_i$$

is finite then $\delta_A E_m{}'' = \delta_A E_m{}' = \delta_A(y_m), m \in \mathbb{N}$ let

$$E = \mathbb{N} \setminus \bigcup_{n=1}^{\infty} E_m{}'$$

if $E$ is finite then put $E_1 = E \cup E_1{}''$ and $E_m = E'_m, n \geq z.......$ if the let $E$ is infinite, then enumerate it as $\{n_1, n_2..........\}$ and put $E_m = E_m{}'' \cup \{n_m\}$. Clearly $\lim_{n \in E_m} m_n = y_m$.

□

**Lemma 2.0.8.** Assume that $\{E_n : n1, 2, 3.......\}$ is partition of $\mathbb{N}$ such that $\sum_{n=1}^{\infty} \delta_A(E_n) < 1$ Then there is partition $\{F_n : n1, 2, 3.......\}$ of $\mathbb{N}$ such that (i)$F_n \subset E_n$;

(ii)$\delta_A(F_n) = \delta_A(E_n)$;

(iii)$\delta_A(F_0) = \sum_{n=1}^{\infty} \delta_A(E_n)$

*Proof.* let $(\delta_n)$ be a strictly decreasing sequence of positive real numbers with $\lim_{n \to \infty} \epsilon_n = 0$.We define inductively a strictly increasing sequence $(n_n)$ of natural numbers such that

$\sum_{k \in [m_{n-1}, j] \setminus (E_1 \cup E_2 \cup ....... \cup E_n)} a_{j,k} \geq 1 - (d(E_1) + d(E_2) + ..... + d(E_n)) - \epsilon_n$ for every $j \geq m_n$.Let

$$F_0 = \bigcup_{n=1}^{\infty} ([m_{n-1}, m_{n+1}] \setminus \bigcup_{i=1}^{n} E_i)$$

where $m_0 = 0$. let $m_n \leqslant j \leqslant m_{n+1}$.

$$\sum_{k\in F_0} a_{j,k} \geq \sum_{k} \in [m_{n-1}, j]\setminus(E_1 \cup E_2 \cup ....... \cup E_n)a_{j,k} \geq 1-(d(E_1)+d(E_2)+.....+d(E_n))-\epsilon_n$$

$$Therefore\ \underline{\delta}_A(F_0) \geq 1 - \sum_{n=1}^{\infty}\delta_A E_n \tag{2.1}$$

let $F_n = E_n \setminus F_0$ for every $n \in \mathbb{N}$.Therefore $\delta_A(F_n) = \delta_A(E_n)$.Since $F_0 = \mathbb{N}\setminus \bigcup_{n=1}^{\infty} F_n$ then

$$\bar{\delta}_A(F_0) \leqslant 1 - \sum_{n=1}^{\infty}\delta_A E_n \tag{2.2}$$

so from equation (2.1) and (2.2) we concluded $\delta_A(F_0) \leqslant 1 - \sum_{n=1}^{\infty}\delta_A E_n$ □

The next theorem gives the sufficient condition for a sequence $(x_n)$ to have $\sum_{y\in D} d_{(y)}$ for some countable set $\mathbb{D}$.

**Theorem 2.0.9.** *Let $(x_n) \in l^{\infty}$ suppose that the set of limit points $(x_n)$ is countable $\delta_A(y)$ exists for any $y \in \mathbb{R}$ Then $\sum_{y\in D} \delta_A(y) = 1$.where $D = \{y \in \mathbb{R} : \delta_A(y) > 0\}$.*

*Proof.* suppose that

$$\sum_{y\in D} \delta_A(y) < 1$$

then by applying proposition with a special case of the form let $[a, b]$ be a fixed interval $a_1$ and $I \in \mathbb{I}$ and point $y \in (a, b)$ is not an $I$-limit points of $(x_n)$.Then the set of limit points of$(x_n)$ in$[a, b]$ is uncountable and closed so the set of $D$ is non-empty by Lemma 1.3.1 the set $D$ is countable.Enumerate $D$ is $(y_1), y_2, ......\}$ by proposition 2.0.7 there is a partition $\{E_k : k = 1, 2, 3......\}$ of $\mathbb{N}$ such that $d(E_k) = d(y_k)$ and

$$\lim_{n\to\infty, n\in E_k} x_n = y_k.$$

By lemma 2.0.8 there is a partition $\{F_k : k = 1, 2, 3......\}$ such that $F_k \subset E_k$,$\delta_A(F_k) = \delta_A(E_k)$ and $\delta_A(F_0 1 - \sum_{k=1}^{\infty} \delta_A(F_k)$. Since $\delta_A(y) = 0$ for every $y \in \mathbb{D}$, then by proposition ( 1.4.4) to the sequence $(x_n)_n \in F_0$ and to the ideal $I_A \mid F_0 = \{E \cap F_0 : E \in I_A\}$ we obtain that the sequence $(x_n)_n \in F_0$ has uncountable many limit points which contradicts the assumptions. □

**Theorem 2.0.10.** ***(Henstock)****[1] Let $(x_n) \in l^{\infty}$.Assume that $A$ is non-negative regular summability matrix. Assume that $G(t) = \delta_A(\{n : x_n \leqslant t)$ exists for every $t \in \mathbb{R}$, then*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} x_k = \int_{-\infty}^{\infty} t dG(t)$$

*Proof.* At first we will show that $G_t = \delta_A(\{n : x_n \leqslant t)$ exists for every $t \in \mathbb{R}$.Fix $\epsilon > 0$. Let $k \in \mathbb{N}$ be such that $\sum_{i=1}^{k} \delta_A(E_i) > 1 - \epsilon$. Let $t \in \mathbb{R} \setminus \{y_1.......y_k\}$ we may assume that $y_1 < ....... < y_k$. Then for any $t$ there is $j = 1, ....., k-1$ such that $y_j < t < y_{j-1}$ or $t < y_1$ or $t < y_k$ Assume that $y_j < t < y_{j-1}$ Put

$$\underline{\delta}_A(E) = \lim_{n\to\infty} \inf \sum_{n\in E} a_{n,k}$$

and

$$\bar{\delta}_A(E) = \lim_{n\to\infty} \sup \sum_{n\in E} a_{n,k}$$

Then

$$\underline{\delta}_A(\{n : x_n \leqslant t\})) \geq dE_1 + ...... + dE_j$$

and

$$\bar{\delta}_A(\{n : x_n \leqslant t\})) \leqslant 1 - d(E_{j+1}, .........-d_{E_k}$$

Thus

$$\bar{\delta}_A(\{n : x_n \leqslant t\})) - \underline{\delta}_A(\{n : x_n \leqslant t\})) \leqslant \epsilon$$

Therefore

$$\delta_A(\{n : x_n \leqslant t\})$$

exists for every $t \in \mathbb{R}$ to prove that

$$\lim_{-\infty}^{\infty} t dt_1(t)$$

$$= \sum_i y_i \delta_A(E_i)$$

we need to show that

$$\lim_{t \to y_i^+} G_{(t)} - \lim_{t \to y_i^-} G_{(t)} = \delta_A(E_i).$$

In fact it is enough to show that

$$\lim_{t \to y_i^+} G_{(t)} - \lim_{t \to y_i^-} G_{(t)} \geq \delta_A(E_i).$$

since $\lim_{t\to\infty} G_t = 1$, $\lim_{t\to\infty} G_t = 0$, $G$ is non-decreasing and $\sum_i \delta_A(E_i) = 1$.

For any $\epsilon > 0$, we have $E_i \diagdown F_\epsilon \in \{n : y_i - \epsilon < x_n < y_i + \epsilon\}$ where $F_\epsilon$ is a finite subset of $\mathbb{N}$. Thus

$$\bar{\delta}_A(\{n : y_i - \epsilon < x_n < y_i + \epsilon\}) \geq \delta_A E_i$$

for any $\epsilon > 0$

Also

$$\begin{aligned} \bar{\delta}_A\{n : y_i - \epsilon < x_n < y_i + \epsilon\} \\ = G(y_i + \epsilon) - G(y_i - \epsilon) \\ = \delta_A\{n : y_i - \epsilon < x_n < y_i + \epsilon\}. \end{aligned}$$

Thus $\lim_{t \to y_i^+} G(t) - \lim_{t \to y_i^-} G(t) \geq \delta_A E_i$. □

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